Countermeasure Policies to Mitigate Random Disruptions in a Capacitated System

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July 24, 2007

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†This work was supported by National Science Foundation Grant DMI 0621030
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Abstract

In this manuscript, we examine a capacitated system exposed to random disruptions with exponentially distributed interarrival time and uniformly distributed magnitude. Each disruption renders a stepwise partial system capacity loss accumulating over time until the remaining capacity reaches a certain level, upon which the system instantaneously regains all lost capacity to the target level. Examples of such a capacity dynamics include shortage of repair personnel and performance degradation caused by failing equipment with a full repair upon a complete failure, non-self-announcing stepwise system failures, and a gradual equipment phaseout and modernization. We explore two different policies for a decision maker who seeks to maximize long-run average reward. Our investigation supported by extensive numerical analysis reveals that reducing the capacity disruption rate is beneficial when certain two-tier policies are engaged in a part of each regenerative cycle. We determine the optimal time to halt countermeasures till the end of the cycle. Our sensitivity study shows that as the system profitability increases and costs of countermeasures are reduced, the optimal policy becomes insensitive to changes in system parameters.

Key Words: capacity analysis, capacity risk management, resource management
1 Introduction

Lean manufacturing philosophy and associated business practices have been widely embraced and deployed by global enterprises. Some estimates assert that the shift to JIT scheduling in the US automotive industry has saved companies more than $1 billion a year in inventory costs, alone. While lean manufacturing has substantially boosted operational efficiency, it has also left enterprises operating in an increasingly risk-encumbered environment. Capacity disruptions triggered by forces of nature, property- and process-related hazards, and man-made interventions have proved to be the most profound influence on enterprise risk. As evidenced in 1995, an earthquake hit the port town of Kobe, Japan, razed to the ground 100,000 buildings and shut down Japan’s largest port for over two years. In 1999, an earthquake in Taiwan displaced power lines to the semiconductor fabrication facilities responsible for more than 50 percent of the worldwide supplies of certain computer components, and shaved 5 percent off earnings for major hardware manufacturers including Dell, Apple, Hewlett-Packard, IBM, and Compaq (Wilcox 1999). In September 2002, longshoremen on the US West Coast were locked out in a labor strike for 11 days, forcing the shutdown of 29 ports. With more than $300 billion of dollars in goods shipped annually through these ports, the dispute caused between $11 and $22 billion in lost sales, spoiled perishables and underutilized capacity (Isidore 2002). In December 2002, a political strike in Venezuela made transnational businesses including GM, BP, Ford, Goodyear and Procter & Gamble halt their manufacturing for the duration of the conflict (Wilson 2003). The recent 2003 outbreak of SARS in China and Singapore forced Motorola to close several plants (Berniker 2003). Man-
made disasters are on the rise, from terrorist attacks to computer viruses (Lemos 2003a, Lemos 2003b). As a result of the above events, according to a recent survey by A.M. Best Company, Inc. of 600 executives, 69 percent of chief financial officers, treasurers and risk managers at Global 1,000 companies in North America and Europe view property-related hazards—such as fires and explosions—and supply chain disruptions as the leading threats to top revenue sources (A.M. Best Company 2006).

Historically, enterprises have lacked appropriate decision support methodologies and computational tools suitable for addressing risk incurred through capacity disruptions. In academia, traditional research efforts on minimizing the cost of supply chain operations and the focus on leveraging economies of scale often yield results that overconcentrate resources. Such optimal solutions can be very sensitive to parameter fluctuations caused by supply chain disruptions. The inability to recognize the hidden costs of such overconcentration heightens the risk of increased costs and capacity imbalance. Much of the recent literature focuses on minimizing costs of supply chain operations (see, for example, Barnes-Schuster et al. 2002, Cheung and Lee 2002, Milner and Kouvelis 2002, Corbett and DeCroix 2001, Lee et al. 1997) whereas only a small fraction of the efforts have been dedicated to modeling the impact of various disruptions such as those affecting demand patterns, supplier and production lead times, prices, imperfect process quality, process yield, and other critical factors.

One of the most common types of disruption appearing in the literature is that of supply rate changes. An excellent work by Arreola-Risa and DeCroix (1998) explores inventory management of stochastic demand systems, where the product supply is disrupted for periods of random duration. The classic economic order
quantity (EOQ) problem with supply disruptions is studied by Parlar and Berkin (1991), and Parlar and Perry (1996) consider order-quantity/reorder-point inventory models with two suppliers subject to independent disruptions to compute the exact form of the average cost expression. Mohebbi (2003) presents an analytical model for computing the stationary distribution of the on-hand inventory in a continuous-review inventory system with compound Poisson demand, Erlang distributed lead time, and lost sales, where the supplier can assume one of the two “available” and “unavailable” states at any point in time according to a continuous-time Markov chain. Papers addressing both supply disruptions and random demand include Chao (1987), Parlar (1997), Song and Zipkin (1996). Chao (1987) proposes a dynamic model concerning optimal inventory policies in the presence of market disruptions, which are often characterized by events with uncertain arrival time, severity and duration. Parlar (1997) considers a continuous-review stochastic inventory problem with random demand and random lead-time where supply may be disrupted due to machine breakdowns, strikes or other randomly occurring events. Song and Zipkin (1996) explore an inventory-control model which includes a detailed Markovian model of the resupply system. A number of papers which address supply and demand changes have been developed in the field of oil stockpiling, as there has been a grave concern over the oil supply from the Middle East (Teisberg 1981, Chap and Manne 1982, Murphy et al. 1987). Modeling production rate disruptions (machine failures) has been largely addressed by extending classical economic manufacturing quantity (EMQ) models. Rosenblatt and Lee (1986) derive an EMQ model when the production process is subject to a random deterioration from an in-control state to an out-of-control state. Lee (1992) models the defect-generating process in the semi-
conductor wafer probe process to determine an optimal lot size, which reduces the average processing time on a critical resource. Abboud (1997) presents a simple approximation of the EMQ model with Poisson machine breakdowns and a low failure rate. Groenvelt et al. (1992b) study an unreliable production system with constant demand and random breakdowns, with a focus on the effects of machine failure and repair on optimal lot sizing decisions. Assuming exponentially distributed time between failures and instantaneous repair of the machine, authors derive some unique properties of their model compared to the classical EMQ model. Groenvelt et al. (1992a) extend their earlier work in Groenvelt et al. (1992b) to the case where repair times are randomly distributed and excess demand is lost. Kim and Hong (1997) propose an extension to the model in Groenvelt (1992a), which determines an optimal lot size when a machine is subject to random failures and the time to repair is constant. They formulate average cost functions for the optimal lot size, and derive conditions for determining the optimal lot size. Hopp et al. (1989) present a model that assumes the $(s, S)$ control policy. With Poisson failures and exponential repair times, a cost function is derived. Rahim (1994) presents an integrated model for determining an economic manufacturing quantity, inspection schedule and a control chart design of an imperfect production process, where he assumes that the process is subject to occurrence of a non-Markovian shock having an increasing failure rate. Among other notable examples of such studies are Henig and Gerchak (1990) and Buzacott and Shanthikumar (1993). Finally, Abboud (2001) examines a single machine production and inventory system with a deterministic production and demand rate, when the machine is subject to random failures. The author models the production/inventory system as a Markov chain and develops an algorithm to compute
the potentials that are used to formulate the cost function.

In sum, research efforts addressing the disruption of supply are still comparatively new and scant. Most of the open literature considering various types of disruptions focuses on issues of inventory, ordering, production lot sizing, production scheduling, and management of inventory, setup, and backorder costs. To the best of our knowledge, there have been no attempts to consider introducing countermeasure policies for mitigating unpredicted capacity disruptions in a capacitated system, and analyze the benefits of such policies for the system manager, accounting for his risk attitude. In this paper, we offer a modeling paradigm suitable for capturing the stochastic dynamics of capacity in a capacitated system exposed to hazardous events with emphasis on countermeasures reducing the capacity disruption rate. In a capacitated system where capacity level follows a certain regenerative pattern, we seek to find optimal fraction of time countermeasures should be activated in each system cycle. In addition, we analyze the sensitivity of our results to crucial system attributes such as cost and effectiveness of countermeasures as well as maximum capacity and unit profit.

The rest of the paper is organized as follows. In Section 2, we introduce the necessary notation and problem definition. Section 3 presents an analysis of a one-tier policy where a decision maker considers whether to activate countermeasures during the entirety of a regenerative system cycle or not. A richer class of countermeasure policies is explored in Section 4 where, in particular, we aim to maximize the long-run average return to obtain the best mitigation policy. In Section 5, we present a numerical analysis of system behavior to understand the sensitivity of optimal policy to changes in various parameter values. Section 6 offers the concluding
2 Notation and problem definition

For the rest of this paper, we define *throughput* as the long-run average of the number of item units per unit time processed by a capacitated system, and the *available system capacity at time* $t$, $C_t$, is defined as the maximum throughput that system resources are capable of sustaining at $t$. Consider a system with a maximum (target) capacity $C^*$ experiencing periodic random disruptions, each of which may render a full or partial system capacity loss. We assume that disruptions occur one at a time and that the $i^{th}$ occurrence results in an instantaneous loss of magnitude $\Delta C_i$ in the remaining system capacity. Following the $i^{th}$ disruption at time $t$, the system capacity remains at level $C_t - \Delta C_i$ until the next disruption unless the remaining capacity falls below a critical level $c$ upon which the system instantaneously regains all lost capacity back to $C^*$. The system assumed to stochastically regenerate at points of recovery (Fig. 1). Capacity dynamics as such can be observed in a number of industrial scenarios including, but are not limited to, shortage of repair personnel and performance degradation caused by failing equipment with a full repair upon a complete failure, non-self-announcing stepwise system failures, and a gradual equipment phaseout and modernization.

Let $\Delta C_i = \alpha_i C^*$, where $\{0 \leq \alpha_i \leq 1, i \in \mathbb{N}\}$ are assumed to form a sequence of i.i.d. random variables. The time of the first disruption is denoted by $X_1$, and $X_i, i = 2, 3, \ldots$ denotes the time between $(i - 1)^{th}$ and $i^{th}$ disruptions (Fig. 1). We assume that $\{X_i, i = 1, 2, \ldots\}$ are i.i.d. random variables. The time of the $n^{th}$
capacity loss is expressed as $Z_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, ...$ where we define $Z_0 = 0$. Let $N_x = \min\{n \text{ s.t. } \sum_{i=1}^{n} \Delta C_i > C^* - x\}$. It then follows that $N_c$ is the number of capacity disruptions between two successive recovery epochs. As such, $Y = Z_{N_c}$ is the time between two successive recovery events, which marks the beginning and the end of one cycle in the regenerative process.

A proactive decision maker has a number of countermeasure policy options to reduce the rate of disruptions. When no countermeasures are activated, he earns $R_t = \pi \cdot C_t$ at time $t$, where $\pi$ is a time independent price factor minus item unit cost. Therefore, revenue in each cycle is $R = \pi \cdot \int_0^Y C_t dt = \pi \cdot C$. A cost of $c(\lambda)$ per unit time is incurred to activate and operate a set of countermeasures that would maintain a rate of $\lambda$ capacity disruptions per unit time. The decision maker’s objective is to select $\lambda$ that maximizes long-run average reward. In this paper, we first consider the case in which countermeasures are activated during the entire cycle. Let $\Pi = R - c(\lambda) \cdot Y$ denote the total reward earned in one renewal cycle,
and $\Pi_t = \int_0^t R_t \, dt - c(\lambda) \cdot t$ denote the total reward earned by time $t$. Then using the result of Theorem 3.6.1 in Ross (1996), the long-run average reward converges to

$$\frac{\Pi_t}{t} \to \frac{E(\Pi)}{E(Y)}.$$ 

Since $E(\Pi) = \pi \cdot E(C) - c(\lambda) \cdot E(Y)$, in the next section we seek to derive the expected cycle length and the expected cycle capacity to obtain the expected reward.

### 3 Long-run average reward

In this section, we assume that interarrival times $X_i$ are distributed exponentially with rate $\lambda$ and that fractional capacity losses $\alpha_i$ are distributed uniformly over $[0,1]$. Total capacity per cycle can be expressed as

$$C = C^* \left[ \sum_{i=1}^{N_c} X_i - \sum_{i=2}^{N_c} \sum_{j=1}^{i-1} \alpha_j X_i \right],$$

whereas the cycle length is $Y = \sum_{i=1}^{N_c} X_i$. Before we proceed with computing $E(C)$ and $E(Y)$, we will need the following result to compute $E(N_c)$, the expected number of capacity loss events per cycle.

**Result 1** Let $\zeta_i, i = 1, \ldots, n$ be i.i.d. random variables distributed uniformly over $[0,1]$. Then $P(\sum_{i=1}^{n} \zeta_i \leq u) = \frac{u^n}{n!}$.

**Proof:** We prove by induction. For $n = 1$, the result is trivial. Assuming that the result holds for $n - 1$, note that $f_{\zeta_1, \zeta_2, \ldots, \zeta_{n-1}}(u) = \frac{u^{n-2}}{(n-2)!}$. We have
\[
P(\sum_{i=1}^{n} \zeta_i \leq u) = P(\sum_{i=1}^{n-1} \zeta_i + \zeta_n \leq u) \\
= \int_{0}^{u} \int_{0}^{u-s} dy \frac{s^{n-2}}{(n-2)!} ds = \int_{0}^{u} (u-s) \frac{s^{n-2}}{(n-2)!} ds \\
= \frac{u^n}{(n-1)!} - \frac{u^n}{n(n-2)!} = \frac{u^n}{n!}. \]

Now we are in a position to compute \( E(Y) \) using Result 1. Let \( c = (1-\alpha)C^* \) for some \( \alpha \in (0,1) \). Note the equivalency of events \( \{N_c = n\} \) and \( \left\{ \sum_{i=1}^{n-1} \alpha_i \leq \alpha \text{ and } \sum_{i=1}^{n} \alpha_i > \alpha \right\} \). Using Result 1 we have,

\[
P(\sum_{i=1}^{n-1} \alpha_i \leq \alpha \text{ and } \sum_{i=1}^{n} \alpha_i > \alpha) = P(\sum_{i=1}^{n-1} \alpha_i \leq \alpha) - P(\sum_{i=1}^{n} \alpha_i \leq \alpha) = \frac{\alpha^{n-1}}{(n-1)!} - \frac{\alpha^n}{n!},
\]

which can be used to obtain,

\[
E(N_c) = \sum_{n=1}^{\infty} n \cdot [\frac{\alpha^{n-1}}{(n-1)!} - \frac{\alpha^n}{n!}] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} = exp(\alpha), \tag{2}
\]

and, therefore,

\[
E(Y) = \sum_{n=1}^{\infty} E(\sum_{i=1}^{N_c} X_i | N_c = n) \cdot P(N_c = n) \\
= \sum_{n=1}^{\infty} n \cdot \frac{1}{\lambda} \cdot P(N_c = n) = \frac{1}{\lambda} \cdot E(N_c) = \frac{exp(\alpha)}{\lambda}.
\]
Computation of $E(C)$ can be found in the Appendix. We have that

$$E(C) = \frac{C^*}{\lambda} \left[ \sum_{n=3}^{\infty} P(N_c = n) \cdot \left\{ n - \sum_{k=1}^{n-2} h_k(n, \alpha) - h(n, \alpha) \right\} + P(N_c = 2) \cdot (2 - h(2, \alpha)) + (1 - \alpha) \right], \quad (3)$$

where

$$h_k(n, \alpha) = \frac{k}{(n-k) \cdot \left( \frac{n}{n+1} - 1 \right)} \cdot \left[ n - k - \alpha + \frac{k+1}{n+1} \right],$$

and

$$h(n, \alpha) = \frac{n(n-1)}{n \alpha^{n-1} - \alpha^n} \left[ (1 - \alpha) \frac{\alpha^n}{n} + \frac{\alpha^{n+1}}{n+1} \right].$$

Using (3), we can compute the long-run average reward in the following way. Define $\bar{C}(\alpha)$ as

$$\bar{C}(\alpha) = \sum_{n=3}^{\infty} P(N_c = n) \cdot \left\{ n - \sum_{k=1}^{n-2} h_k(n, \alpha) - h(n, \alpha) \right\} + P(N_c = 2) \cdot (2 - h(2, \alpha)) + (1 - \alpha).$$

Then, the limiting value of the long-run average reward is given by the following expression.

$$\frac{\Pi_t}{t} \rightarrow \frac{E(\Pi)}{E(Y)} = \frac{\pi \cdot C^* \cdot C(\alpha)}{\lambda} = \frac{c(\lambda)}{\lambda} = \frac{\pi \cdot C^* \cdot C(\alpha)}{\lambda} = \frac{c(\lambda)}{\lambda} - c(\lambda).$$

At first, this result might seem counterintuitive. It implies that the policy to activate no countermeasures (where $c(\cdot) = 0$) is superior to the policy of activating
countermeasures during the entirety of the regenerative cycle. The disruption rate \( \lambda \) has no impact on the long-run average revenue because the capacity recovery decisions are independent of the cycle length. A positive impact of reduced disruption rate is a decreased time between capacity losses when the system is maintaining a capacity level close to \( C^* \). However, the system may experience prolonged periods of low capacity levels as well. When the system operates at a level just above \( c \), more frequent disruptions may, in fact, be preferred in order to reach the critical level more rapidly and gain an immediate full recovery. For the capacity dynamics described in this model, these two effects offset each other and no costly countermeasures should be activated to maximize the long-run average reward. However, as we will see in the next section, the policy of activating no countermeasures fails to be optimal among a richer set of policies.

### 4 A two-tier level disruption mitigation policy

Consider the set of policies under which countermeasures are activated at the beginning of each system cycle and remain in effect as long as system capacity exceeds a certain higher level \( c_l > c \), where \( c_l = (1 - \alpha_l) \cdot C^* \) for some \( \alpha_l \in (0, 1) \). Countermeasures remain deactivated for levels below \( c_l \) where the system becomes exposed to “normal” disruption rate. As in Section 2 and 3, \( c \) is the critical lower level that triggers instantaneous capacity recovery (Fig. 2). The system is said to be “on” when countermeasures are in effect and “off” otherwise. Long-run average reward of this altered process exhibits the same convergence property. Therefore, it is our interest in this section to compute \( E[\Pi]/E[Y] \), where in this case
The capacity of the system is given by:

$$\Pi = R - c(\lambda) \cdot \int_0^V 1\{\text{countermeasures are activated}\}(t)dt.$$  

Figure 2: A realization of the system capacity dynamics for a two-level mitigation policy. Disruption rate is reduced for $c_l \leq C_t \leq C^*$. 

We first derive the distribution of the initial system capacity for the “off” period in a cycle. The following proposition summarizes the result,

**Proposition 1** Consider a capacitated system in which capacity disruption interarrival times are exponentially distributed with parameter $\lambda$, fractional stepwise capacity losses follow a uniform distribution on $[0,1]$, and capacity is restored fully and instantaneously upon falling below level $c$. Suppose that the system is “on” when $C_t > (1 - \alpha_l) \cdot C^*$, “off” otherwise, and $c = (1 - \alpha) \cdot C^*$, $\alpha_l < \alpha$. Then, the distribution of initial system capacity of the “off” period is given by the following,

$$P\left( \sum_{i=1}^{N_{cl}} \Delta C_i \leq \bar{\alpha} C^* \right) = (\bar{\alpha} - \alpha_l) \cdot \exp(\alpha_l).$$

**Proof:** We proceed by considering the number of capacity losses during the “on”
period, \(N_{cl}\). Let \(\Gamma_k = \sum_1^k \alpha_i\). For \(n \geq 2,\)

\[
P\left(\sum_{i=1}^{N_{cl}} \Delta C_i \leq \bar{\alpha}C^* | N_{cl} = n \right) = P(\Gamma_{n-1} \leq \alpha_l, \Gamma_n \in (\alpha_l, \bar{\alpha}) | \Gamma_{n-1} \leq \alpha_l, \Gamma_n > \alpha_l)
= P(\Gamma_{n-1} \leq \alpha_l, \Gamma_n \in (\alpha_l, \bar{\alpha})) / P(\Gamma_{n-1} \leq \alpha_l, \Gamma_n > \alpha_l)
= \left[\int_0^{\alpha_l} f_{\Gamma_{n-1}}(s) \cdot P(\alpha_n \in (\alpha_l - s, \bar{\alpha} - s)) ds \right] / \left[\frac{\alpha_l^{n-1}}{(n-1)!} - \frac{\alpha_l^n}{n!}\right]
= (\bar{\alpha} - \alpha_l) \frac{\alpha_l^{n-1}}{(n-1)!} / \left[\frac{\alpha_l^{n-1}}{(n-1)!} - \frac{\alpha_l^n}{n!}\right].
\]

Note that \(P(\alpha_1 \in (\alpha_l, \bar{\alpha}) | \alpha_1 > \alpha_l) = (\bar{\alpha} - \alpha_l) / (1 - \alpha_l)\). Therefore, one can verify by slight modifications in the computations above that (4) holds for \(n = 1\) as well.

Using Result 1, we obtain

\[
P\left(\sum_{i=1}^{N_{cl}} \Delta C_i \leq \bar{\alpha}C^* \right) = (\bar{\alpha} - \alpha_l) \cdot \sum_{n=1}^{\infty} P(N_{cl} = n) \left\{\frac{\alpha_l^{n-1}}{(n-1)!} / \left[\frac{\alpha_l^{n-1}}{(n-1)!} - \frac{\alpha_l^n}{n!}\right]\right\}
= (\bar{\alpha} - \alpha_l) \sum_{n=1}^{\infty} \frac{\alpha_l^{n-1}}{(n-1)!} = (\bar{\alpha} - \alpha_l) \cdot exp(\alpha_l),
\]

which concludes the proof. □

Expected cycle reward is the sum of expected rewards of the “on” and “off” cycle periods. Let \(Y_t\) and \(Y_s\) denote the length of the “on” and “off” cycle periods, respectively, where \(Y = Y_t + Y_s\) is the length of the cycle. Define

\[
\bar{C}_l(\alpha_l) = \sum_{n=3}^{\infty} P(N_{cl} = n) \cdot \left\{n - \sum_{k=1}^{n-2} h_k(n, \alpha_l) - h(n, \alpha_l)\right\}
+ P(N_{cl} = 2) \cdot (2 - h(2, \alpha_l)) + (1 - \alpha_l).
\]
Results of the previous section can be readily applied to obtain the expression for expected total capacity, $C_t$, during the “on” period, which is $E(C_t) = \bar{C}_t(\alpha_t) \cdot C^*/\lambda_t$, where $\lambda_t$ is disruption rate during the “on” period. Similarly, we have $E(Y_t) = \text{exp}(\alpha_t)/\lambda_t$. While the length of each cycle is affected by the change in disruption rate, the total number of disruption events in a cycle, $N_c$, is determined solely by a uniform capacity reduction process that evolves independently from the disruption rate. Therefore, we can deduce immediately using (2) that $E(N_c) = \text{exp}(\alpha)$ and $E(N_{cl}) = \text{exp}(\alpha_t)$. Then, since capacity disruption rate remains at $\lambda$ during the “off” period we have,

$$E(Y_s) = \frac{\text{exp}(\alpha) - \text{exp}(\alpha_t)}{\lambda}.$$  

Likewise, expected total capacity during the “off” period, $C_s$, can be readily obtained after considering the initial capacity level in the “off” period, $C_0$. Note that

$$C_s = C_0 \cdot \sum_{i=1}^{N_c-N_{cl}} X_i - C^* \cdot \sum_{i=2}^{N_c-N_{cl}} \sum_{j=1}^{i-1} X_i \alpha_j,$$

so that

$$E(C_s|C_0 = (1 - \bar{\alpha}) \cdot C^*) = C_0 \cdot \frac{E(N_c - N_{cl}|C_0 = (1 - \bar{\alpha}) \cdot C^*)}{\lambda}$$

$$- \frac{C^*}{\lambda} \cdot \sum_{n=2}^{\infty} P(N_c - N_{cl} = n|C_0 = (1 - \bar{\alpha}) \cdot C^*) \cdot \sum_{i=2}^{n} E\left(\sum_{j=1}^{i-1} \alpha_j |N_c - N_{cl} = n\right).$$

This expression can be simplified to yield the following:
\[
E(C_s|C_0) = \frac{C_0}{\lambda} \cdot \exp(\alpha - \bar{\alpha}) - \frac{C^*}{\lambda} \cdot \left[\sum_{n=3}^{\infty} P(N_c - N_{cl} = n \mid C_0 = (1 - \bar{\alpha}) \cdot C^*) \cdot \left\{ \sum_{k=1}^{n-2} h_k(n, \alpha - \bar{\alpha}) + h(n, \alpha - \bar{\alpha}) \right\} + P(N_c - N_{cl} = 2 \mid C_0 = (1 - \bar{\alpha}) \cdot C^*) \cdot h(2, \alpha - \bar{\alpha}) \right]
\]
\[
= \frac{C_0}{\lambda} \cdot \exp(\alpha - \bar{\alpha}) - \frac{C^*}{\lambda} \cdot \psi(\bar{\alpha}, \alpha).
\]

We also have
\[
P(N_c - N_{cl} = n \mid C_0 = (1 - \bar{\alpha}) \cdot C^*) = \frac{(\alpha - \bar{\alpha})^n}{(n - 1)!} - \frac{(\alpha - \bar{\alpha})^n}{n!}.
\]

In order to compute \(E(C_s)\), we need the expression for \(E(C_0 \cdot \exp(\alpha - \bar{\alpha}))\), which is derived as follows.

\[
E(C_0 \cdot \exp(\alpha - \bar{\alpha})) = E(E(C_0 \cdot \exp(\alpha - \bar{\alpha})|N_{cl}))
\]
\[
= C^* \cdot \sum_{n=1}^{\infty} P(N_{cl} = n) \cdot \int_{\alpha_l}^{\alpha} (1 - \bar{\alpha}) \cdot \exp(\alpha - \bar{\alpha}) \cdot \left\{ \frac{\alpha_i^{n-1}}{(n - 1)!} \left[ \frac{\alpha_i^{n-1}}{(n - 1)!} - \frac{\alpha_i^n}{n!} \right] \right\} d\bar{\alpha}
\]
\[
= C^* \cdot (\alpha - \alpha_1 \cdot \exp(\alpha - \alpha_1)) \cdot \sum_{n=1}^{\infty} \left[ \frac{\alpha_i^{n-1}}{(n - 1)!} - \frac{\alpha_i^n}{n!} \right] \cdot \left\{ \frac{\alpha_i^{n-1}}{(n - 1)!} - \frac{\alpha_i^n}{n!} \right\}
\]
\[
= C^* \cdot (\alpha - \alpha_1 \cdot \exp(\alpha - \alpha_1)) \cdot \exp(\alpha_1) = C^* \cdot (\alpha \cdot \exp(\alpha_1) - \alpha_1 \cdot \exp(\alpha)).
\]

Now, we are in a position to obtain \(E(C_s)\):
This brings us to the following principle proposition.

**Proposition 2** For the capacitated system described in Proposition 1, the long-run average reward converges to the following expression.

\[
\begin{align*}
\Pi_t & \rightarrow \frac{\pi \cdot (E(C_l) + E(C_s)) - E(Y_l) \cdot c(\lambda_l)}{E(Y_s) + E(Y_l)}. \\
\end{align*}
\]

**Proof:** Using Theorem 3.6.1 in Ross (1996), we know that the long-run average reward converges to,

\[
\begin{align*}
\Pi_t & \rightarrow \frac{\pi \cdot \left(\frac{C^* \cdot C_l(\alpha_l)}{\lambda_l} + \frac{C^* \cdot f(\alpha_l)}{\lambda_l} \right) - \frac{\exp(\alpha_l) \cdot c(\lambda_l)}{\exp(\alpha_l)/\lambda_l + [(\exp(\alpha) - \exp(\alpha_l))/\lambda_l]}}{E(Y_s) + E(Y_l)}. \\
\end{align*}
\]

The proof then follows by substituting expressions for \(E(C_l), E(C_s), E(Y_l)\) and \(E(Y_s)\) into (6). \(\square\)

5 Discussion of computational results

An optimal two-tier policy maximizes long-run average reward expressed in equation (5) by activating countermeasures that set optimal levels of \(\lambda_l\) and \(\alpha_l\). In what follows, we conduct a parametric analysis of the optimal policy behavior for fixed \(\alpha\). Since the disrupted capacity is regained instantaneously and cost-free at the end of
each cycle, the solution to an optimum policy \((\alpha_1^*, \alpha^*)\) is trivial. Hence, we analyze the behavior of the optimal \(\alpha_1\) as a function of \(\alpha\), i.e., \(\alpha_1^*(\alpha)\), for fixed values of \(\lambda\) and \(\lambda_l\). While it is difficult to provide a formal proof, \(\alpha_1^*(\alpha)\) is monotonically non-decreasing in \(\alpha\). To see this, one can note that as \(\alpha\) increases, the cycle time is likely to increase as well, and so is the period of lower system capacity. On the other hand, increasing \(\alpha_l\) results in longer periods of higher system capacity. This trade-off between benefits and costs of activating countermeasures renders \(\alpha_1^*(\alpha) < \alpha\). The plots discussed in this section confirm this intuition. The initial parameter values of our analysis are listed in Table 1.

Table 1: Initial parameter values.

<table>
<thead>
<tr>
<th>(\lambda_l)</th>
<th>0.0005</th>
<th>(\pi)</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>0.001</td>
<td>(C^*)</td>
<td>1</td>
</tr>
<tr>
<td>(r)</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The cost of countermeasures is assumed to be of the form \(c(\lambda_l) = (\lambda/\lambda_l)^r\). The cost decreases as the disruption rate \(\lambda_l\) gets higher, which is used to measure the effectiveness of countermeasure technology, and the cost increases in \(r\), which is used to model the marginal cost of installing a more effective technology. As Figure 3 illustrates, \(\alpha_1^*(\alpha)\) is increasingly decreasing in \(r\). We also observe that as \(r\) gets larger, \(\alpha_1^*(\alpha)\) exhibits a higher sensitivity to per unit changes in \(r\). In flat regions of Figure 3, reducing the disruption rate is not economically sound.

For a linear cost function \((r = 1, \text{Figure 4a})\), \(\alpha_1^*(\alpha)\) is increasing in \((\lambda_l/\lambda)\), which implies that the incremental benefits of reducing the rate of capacity disruptions do not warrant the use of countermeasures over extended periods of time. However, if the marginal cost of installing better countermeasures is not constant \((r = 0.001,\)
Figure 3: A 3-dimensional representation of $\alpha_r^*$ as a function of $\alpha$ and $r$.

Figure 4(b), the plots of $\alpha_r^*(\alpha)$ for different values of $(\lambda_l/\lambda)$ intersect. If the marginal cost of a decreased $(\lambda_l/\lambda)$ is relatively small, then $\alpha_r^*(\alpha)$ may be increasing with a more advanced technology (this relationship does not hold for higher values of $\alpha$). However, both plots agree that the rate of increase of $\alpha_r^*(\alpha)$ is higher in the lower region of $(\lambda_l/\lambda)$ values. Therefore, we see that expected increase in countermeasure costs over extended periods outweighs the benefits of better technology.

Furthermore, we observe that $\alpha_r^*(\alpha)$ is insensitive to changes in maximum capacity $C^*$ in the neighborhood of initial parameter values in Table 1. Common wisdom, however, suggests that $C^*$ shall be positively correlated with the optimum period of activated countermeasures. Should all items be sold, increasing $C^*$ would lead
Figure 4: Behavior of $\alpha_t^*(\alpha)$ for different values of $\lambda_l/\lambda$.

![Behavior of $\alpha_t^*(\alpha)$ for different values of $\lambda_l/\lambda$.](image)

a. $r = 1$

b. $r = 0.001$

to higher profits and hence, countermeasures should be engaged for longer periods (See Figure 5a, where $C^*$ takes values in $[0, 0.1]$). As $C^*$ approaches 0.1, marginal increase in $\alpha_t^*(\alpha)$ falls off sharply. This suggests that the optimal period of activated countermeasures is insensitive to changes in maximum capacity, if $C^*$ is already high. Also, the region of sensitivity of $C^*$ is a function of the unit profit. As illustrated in Figure 5b, $\alpha_t^*(\alpha)$ becomes responsive to changes in $C^* \in [0.1, 1.0]$, when $\pi$ gets smaller (this change in sensitivity may be minimal if $C^*$ is already high).

A similar relation exists between $\alpha_t^*(\alpha)$ and $\pi$. Figure 6 illustrates that $C^*$ is insensitive to changes in $\pi$ around the original parameter value of $\pi = 1000$ whereas at lower unit profit levels, marginal changes in $\pi$ render larger perturbations in $\alpha_t^*(\alpha)$. Changes in system capacity for low value items may require more radical changes in countermeasure policy. Nevertheless, the region of sensitivity is relatively small for both $\pi$ and $C^*$, which suggests on a larger scale that $\alpha_t^*(\alpha)$ is quite robust
Figure 5: Behavior of $\alpha^*_t(\alpha)$ for different values of $\alpha$ and $C^*$.  

da. $\pi = 1000$  

b. $\pi = 80$  

Figure 6: Behavior of $\alpha^*_t(\alpha)$ for different values of $\alpha$ and $\pi$.  

21
to changes in system profitability.

6 Conclusions

Global enterprises hinge upon large and convoluted networks of production facilities, warehouses, transportation systems, and customers which operate under an increasing risk of capacity disruptions of various origin. Facing a fierce competition, many companies are forced to reduce redundancy in their operations and convert them to a lean mode. This elevates the risk of capacity imbalance and business interruptions. In addition to property- and process-related hazards, the ongoing drive to enter emerging markets offers new opportunities, albeit under less familiar dynamics. The research community has just recently begun to address these issues and thus far, the related academic literature is still novel and insufficient. Most efforts have been concentrated on development of somewhat retroactive strategies with a focal point on the issues of inventory strategies, ordering policies, production lot sizing, production scheduling, and management of inventory, setup, and backorder costs. However, as far as our knowledge protracts, there have been no attempts to consider the development of active countermeasure policies for managing capacitated systems in the presence of unpredicted capacity disruptions.

This paper discusses potential impacts of countermeasures for mitigating the risk of hazardous events in a capacitated system experiencing stepwise full or partial system capacity losses of random occurrence and duration, followed by an instantaneous recovery. For such a system where capacity is restored in a regenerative mode, we aimed to find the optimal fraction of time when countermeasures should
be activated in each regenerative cycle. In practice, these countermeasures could range from purely technological solutions, such as installation of fire prevention water sprinkler systems, to non-technological routes that could, for example, alleviate labor strikes or prevent terrorist attacks. We have first found that for exponentially distributed interarrival times and uniformly distributed magnitudes of disruptions, reducing disruption rate provides no benefits to the decision maker when countermeasures are activated during the entirety of a regenerative cycle. This result follows mainly from the nature of the stepwise partial system capacity degradation followed by an instantaneous and cost-free restoration to the maximum level upon reaching a certain critical threshold. Reducing the frequency of disruption events has shown to be somewhat favorable at higher levels of system capacity, but it may also prove to be costly as it may substantially extend low capacity periods in each cycle.

The next section of the manuscript analyzed alternative policies to improve the performance of countermeasures. We considered a class of two-tier mitigation policies for which countermeasures are engaged at the beginning of each cycle and terminated as the cumulative system capacity loss exceeds a certain level. We determined the optimal time to halt countermeasures till the end of the cycle. The long-run average reward computed under the new policies suggests that activating countermeasures early in the cycle is superior to the policy of abandoning all the countermeasures during the entirety of the cycle.

In this paper, we did not address the question of the best critical threshold that initiates immediate capacity recovery, as we assumed that the cost associated with administering any level of $\alpha$ is zero. Therefore, the problem of obtaining the optimal pair $(\alpha^*_I, \alpha^*)$ has a trivial solution (i.e., set $\alpha^* = 0$). Rather, we sought
to find the optimal time in each regenerative cycle when countermeasures should be terminated given a capacity recovery threshold of \((1 - \alpha) \cdot C^*\). Since derivation of closed form expressions for some of these results has proved to be problematic, Section 5 addressed this question using a numerical analysis to determine optimal \(\alpha^*_l(\alpha)\) that maximized long-run average reward under various parametric settings. We presented the results of our sensitivity analysis for an exponential cost function. In general, \(\alpha^*_l(\alpha)\) was found to be quite sensitive to exponentially increasing cost, as well as capacity and unit profit changes, if the system was already operating with low profit margins. However, as the profitability of the system increased, \(\alpha^*_l(\alpha)\) had a robust response to system parameter changes.

The model presented in this paper can be generalized to address similar questions in capacitated systems evolving under different environments dictating more complex capacity dynamics. This paper provides a set of preliminary results for optimizing capacity disruption countermeasure decisions. Development of models that study similar decisions under costly capacity recovery and loss are necessary to improve our understanding on capacity management. We further believe that similar single facility models will form a basis to approach resource allocation problems to counter business interruption risk in large supply chains.

**Appendix**

**Computation of \(E(C)\) in equation (3).** We begin by conditioning on \(N_c\). Noting the dependency between \(\Delta C_i\) and \(N_c\), we first compute the conditional density of \(\Gamma_k = \sum_{i=1}^{k} \alpha_i, f_{\Gamma_k}(s|N_c = n)\) for \(k < n - 1\) and \(n \geq 3\). Note that
\[
f_{\Gamma_k}(s|N_c = n) = \int_s^\alpha P(\Gamma_k = s, \Gamma_{n-1} = u, \Gamma_n > \alpha) \, \frac{P(\Gamma_{n-1} \leq \alpha, \Gamma_n > \alpha)}{P(\Gamma_{n-1} \leq \alpha, \Gamma_n > \alpha)} \, du. \tag{7}
\]

Since \(\alpha_i\)'s are independent, we can use Result 1 to obtain

\[
P(\Gamma_k = s, \Gamma_{n-1} = u, \Gamma_n > \alpha) = P(\sum_{i=1}^{k} \alpha_i = s) \cdot P(\sum_{i=k+1}^{n-1} \alpha_i = u) \cdot P(\alpha_n > \alpha - u)
= (1 + u - \alpha) \cdot \frac{s^{k-1} \cdot (u-s)^{n-k-2}}{(k-1)! \cdot (n-k-2)!}. \tag{8}
\]

Substituting (8) in (7) and evaluating the integral, we obtain

\[
f_{\Gamma_k}(s|N_c = n) = \frac{1}{P(\Gamma_{n-1} \leq \alpha, \Gamma_n > \alpha)} \int_s^\alpha (1 + u - \alpha) \cdot \frac{s^{k-1} \cdot (u-s)^{n-k-2}}{(k-1)! \cdot (n-k-2)!} \, du
= \frac{s^{k-1}}{(k-1)!} \cdot \left[ \frac{(\alpha-s)^{n-k-1}}{(n-k-1)!} - \frac{(\alpha-s)^{n-k}}{(n-k)!} \right] / \left[ \frac{\alpha^{n-1}}{(n-1)!} - \frac{\alpha^n}{n!} \right].
\]

We use this conditional density to compute \(E(\Gamma_k|N_c = n)\) as follows:

\[
E(\Gamma_k|N_c = n) = \int_0^\alpha \frac{s^k}{(k-1)!} \cdot \left[ \frac{(\alpha-s)^{n-k-1}}{(n-k-1)!} - \frac{(\alpha-s)^{n-k}}{(n-k)!} \right] / \left[ \frac{\alpha^{n-1}}{(n-1)!} - \frac{\alpha^n}{n!} \right] \, ds.
= \frac{k \cdot n!}{k!(n-k)!} \cdot \frac{1}{n\alpha^{n-1} - \alpha^n}.
\]
\[(n - k - \alpha) \int_0^\alpha s^k (\alpha - s)^{n-k-1} \, ds + \int_0^\alpha s^{k+1} (\alpha - s)^{n-k-1} \, ds,\]

Note that both integrals represent beta functions (to see this, make a variable change \(t = s/\alpha\)). Therefore, we arrange the expressions to obtain

\[
E(\Gamma_k|N_c = n) = k \frac{n!}{k!(n-k)!} \cdot \frac{1}{n \alpha^{n-1} - \alpha^n} \cdot [(n - k - \alpha)\alpha^n B(k + 1, n - k) + \alpha^{n+1} B(k + 2, n - k)]
\]

\[
= \frac{k}{(n-k) \cdot \left( \frac{n}{\alpha} - 1 \right)} \cdot [n - k - \alpha + \frac{k+1}{n+1}]
\]

\[
= h_k(n, \alpha). \tag{9}\]

Note that this expression holds for \(k < n - 1\) and \(n \geq 3\). For \(k = n - 1\) and \(n \geq 2\),

\[
f_{\Gamma_{n-1}}(s|N_c = n) = \frac{f_{\Gamma_{n-1}}(s) \cdot P(\alpha_n > \alpha - s)}{P(\Gamma_{n-1} \leq \alpha, \Gamma_n > \alpha)}.\]

Using the above expression, we eventually obtain

\[
E(\Gamma_{n-1}|N_c = n) = \frac{1}{\alpha^{n-1}} \cdot \frac{\alpha^n}{n!} \int_0^\alpha s^{n-1} \frac{1}{(n-2)!} \cdot (1 - \alpha + s) \, ds
\]

\[
= \frac{n(n-1)}{n \alpha^{n-1} - \alpha^n} \left[ (1 - \alpha) \frac{\alpha^n}{n} + \frac{\alpha^{n+1}}{n+1} \right]
\]

\[
= h(n, \alpha). \tag{10}\]
We can now derive expected capacity per cycle. Equation (1) gives

\[
E(C) = E\left(C^* \cdot E\left(\sum_{i=1}^{N_c} X_i - \sum_{i=2}^{N_c} \sum_{j=1}^{i-1} X_i \alpha_j | N_c = n\right)\right)
\]

\[
= C^* \left\{ \sum_{n=2}^{\infty} P(N_c = n) \cdot \left\{ \frac{n}{\lambda} - E\left(\sum_{i=2}^{n} \sum_{j=1}^{i-1} X_i \alpha_j | N_c = n\right) \right\} + \frac{P(N_c = 1)}{\lambda} \right\}
\]

\[
= C^* \left\{ \sum_{n=2}^{\infty} P(N_c = n) \cdot \left\{ \frac{n}{\lambda} - \sum_{i=2}^{n} E(\Gamma_i | N_c = n) \right\} \right\}
\]

\[
= \frac{C^*}{\lambda} \left\{ \sum_{n=2}^{\infty} P(N_c = n) \cdot \left\{ n \sum_{i=2}^{n} E(\Gamma_i | N_c = n) \right\} \right\}
\]

Finally, substituting equations (9) and (10), we obtain

\[
E(C) = C^* \left\{ \sum_{n=3}^{\infty} P(N_c = n) \cdot \left\{ n - \sum_{k=1}^{n-2} h_k(n, \alpha) - h(n, \alpha) \right\} \right\}
\]

\[
+ P(N_c = 2) \cdot (2 - h(2, \alpha)) + (1 - \alpha)]
\]

\[
= \frac{C^*}{\lambda} E(N_c) - \frac{C^*}{\lambda} \left\{ \sum_{n=3}^{\infty} P(N_c = n) \cdot \left\{ \sum_{k=1}^{n-2} h_k(n, \alpha) + h(n, \alpha) \right\} + P(N_c = 2) \cdot h(2, \alpha) \right\}
\]

\[
= \frac{C^*}{\lambda} \cdot \exp(\alpha) - \frac{C^*}{\lambda} \left\{ \sum_{n=3}^{\infty} P(N_c = n) \cdot \left\{ \sum_{k=1}^{n-2} h_k(n, \alpha) + h(n, \alpha) \right\} + P(N_c = 2) \cdot h(2, \alpha) \right\}.
\]
References


